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Stable vortex solitons in nonlocal self-focusing nonlinear media

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We reveal that spatially localized vortex solitons become *stable* in self-focusing nonlinear media when the vortex symmetry-breaking azimuthal instability is eliminated by a nonlocal nonlinear response. We study the main properties of different types of vortex beams and discuss the physical mechanism of the vortex stabilization in spatially nonlocal nonlinear media.

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Vortices are fundamental objects which appear in many branches of physics [1]. In optics, vortices are usually associated with phase singularities of diffracting optical beams, and they can be generated experimentally in different types of linear and nonlinear media [2]. However, optical vortices become *highly unstable* in self-focusing nonlinear media due to the symmetry-breaking azimuthal instability, and they decay into several fundamental solitons [3]. In spite of many theoretical ideas to stabilize optical vortices in specific nonlinear media [4], no stable optical vortices created by coherent light were readily observed in experiment [5]. Thus, the important challenge remains to reveal physical mechanisms which would allow experimental observation of stable coherent vortices in realistic nonlinear media.

In this Communication, we reveal that the symmetry-breaking azimuthal instability of the vortex beams can be suppressed and *even eliminated completely* in the media characterized by a nonlocal nonlinear response. This observation allows us to suggest a simple and realistic way to generate experimentally stable, spatially localized vortices in self-focusing nonlinear media. We study the main properties and stability of different types of vortex beams, and discuss the physical mechanism of their stabilization in spatially nonlocal nonlinear media.

We notice that there exist many physical systems characterized by nonlocal nonlinear response. In particular, a nonlocal response is induced by heating and ionization, and it is known to be important in media with thermal nonlinearities such as thermal glass [6] and plasmas [7]. Nonlocal response is a key feature of the orientational nonlinearities due to long-range molecular interactions in nematic liquid crystals [8]. An interatomic interaction potential in Bose-Einstein condensates with dipole-dipole interactions is also known to be substantially nonlocal [9]. In all such systems, nonlocal nonlinearity can be responsible for many unique features such as the familiar effect of the collapse arrest [10,11].

We consider propagation of the electric-field envelope $E(X,Y,Z)$ described by the paraxial wave equation,

$$2ik_0\frac{\partial E}{\partial Z}+\frac{\partial^2 E}{\partial X^2}+\frac{\partial^2 E}{\partial Y^2}+k_0^2 n_T\Theta E=0, \qquad (1)$$

where k_0 is the wave number and the function Θ characterizes a nonlinear, generally nonlocal, medium response. For example, in the case of the wave beam propagation in partially ionized plasmas, $\Theta=T_e'/T$ is the relative electron temperature perturbation, with T being the unperturbed temperature, and the coupling coefficient $n_T=+1$. Stationary temperature perturbation obeys the equation [7],

$$\alpha^2\Theta-l_e^2\left[\frac{\partial^2\Theta}{\partial X^2}+\frac{\partial^2\Theta}{\partial Y^2}+\frac{\partial^2\Theta}{\partial Z^2}\right]=|E|^2/E_c^2, \qquad (2)$$

where $E_c^2=3mT(\omega_0^2+\nu_e^2)/e^2$, ν_e is the electron collision frequency, ω_0 is the wave frequency, m is electron mass, and $\alpha^2\approx 2m/M$ characterizes the relative energy that the electron delivers to a heavy particle with mass M during single collision. The second term describes *thermal diffusion* with the characteristic spatial scale l_e. We mention that the model *identical* to Eqs. (1) and (2) describes the light propagation in media with thermal nonlinearities [6], and it appears also in the study of two-dimensional bright solitons in nematic liquid crystals [8] recently observed in experiment [12]. In the latter case, the field Θ describes the spatial distribution of the molecular director.

Rescaling the variables $(X,Y)=l_e(x,y)$ and $Z=2l_ez/\epsilon$, and the fields, $E=(E_c\epsilon/\sqrt{n_T})\Psi(x,y,z)$ and $\Theta=(\epsilon^2/n_T)\theta(x,y,z)$, where $\epsilon=(k_0l_e)^{-1}$, we present Eq. (2) in the dimensionless form,

$$\alpha^2\theta-\Delta_\perp\theta-\frac{\epsilon^2}{4}\frac{\partial^2\theta}{\partial z^2}=|\Psi|^2, \qquad (3)$$

where $\Delta_\perp=\partial^2/\partial x^2+\partial^2/\partial y^2$ is the transverse Laplacian. For the analysis performed below, we omit in Eq. (3) the term proportional to ϵ^2.

Thus, the basic dimensionless equations describing the propagation of the electric-field envelope $\Psi(x,y,z)$ coupled to the temperature perturbation $\theta(x,y,z)$ become

$$i\frac{\partial\Psi}{\partial z}+\Delta_\perp\Psi+\theta\Psi=0,$$

$$\alpha^2\theta-\Delta_\perp\theta=|\Psi|^2. \qquad (4)$$

In the limit $\alpha^2\gg 1$, we can neglect the second term in the equation for the field θ of Eq. (4) and reduce this system to

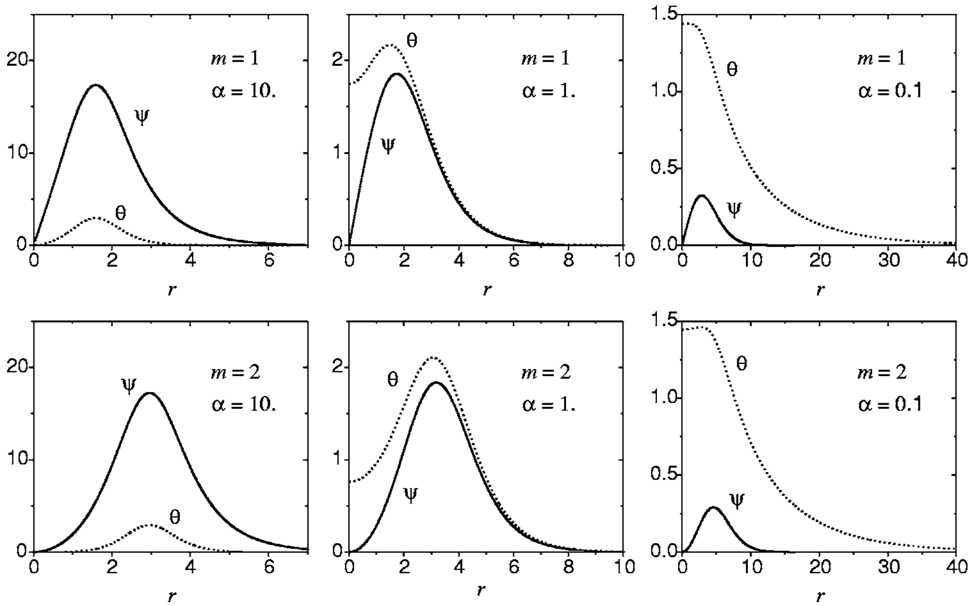

FIG. 1. Examples of the stationary vortex solutions for $m=1$ and $m=2$ at different values of the nonlocality parameter α. Shown are the fields $\psi(r)$ (solid line) and $\theta(r)$ (dotted line).

the standard local nonlinear Schrödinger (NLS) equation with cubic nonlinearity. The opposite case, i.e., $\alpha^2 \ll 1$, will be referred to as a strongly nonlocal regime of the beam propagation.

We look for the stationary solutions of the system (4) in the form $\Psi(x,y,z)=\psi(r)\exp(im\varphi+i\Lambda z)$, where φ and $r=\sqrt{x^2+y^2}$ are the azimuthal angle and the radial coordinate, respectively, and Λ is the beam propagation constant. Such solutions describe either the fundamental optical soliton, when $m=0$, or the vortex soliton with the topological charge m, when $m\neq 0$.

The beam radial profile $\psi(r)$ and the temperature field $\theta(r)$ associated with it can be found by solving the system of ordinary differential equations,

$$-\lambda\psi+\Delta_r^{(m)}\psi+\theta\psi=0;\quad \alpha^2\theta-\Delta_r^{(0)}\theta=|\psi|^2, \tag{5}$$

where $\Delta_r^{(m)}=d^2/dr^2+(1/r)(d/dr)-(m^2/r^2)$, and $\psi,\theta,1/r^2,\alpha^2$ are rescaled by the factor of the propagation constant Λ which itself becomes $\Lambda=1$. Boundary conditions are: for the localized vortex field, $\psi(\infty)=\psi(0)=0$, and for the temperature field, $d\theta/dr|_{r=0}=0$ and $\theta(\infty)=0$.

The second equation of the system (4) can be readily solved for radially symmetric intensity distribution $|\Psi|^2$,

$$\theta(r,z)=\int_0^{+\infty}|\Psi(\xi,z)|^2G_0(r,\xi;\alpha)\xi d\xi, \tag{6}$$

where G_0 is the Green's function defined at $\nu=0$ from the general expression

$$G_\nu(\xi_1,\xi_2;a)=\begin{cases}K_\nu(a\xi_2)I_\nu(a\xi_1), & 0\leq\xi_1<\xi_2,\\ I_\nu(a\xi_2)K_\nu(a\xi_1), & \xi_2<\xi_1<+\infty,\end{cases} \tag{7}$$

and I_ν and K_ν are the modified Bessel functions of the first and second kind, respectively. Thus, Eqs. (5) are equivalent to a single integrodifferential equation obtained from Eqs. (5) when the function $\theta(r)$ is eliminated, or to a single integral equation,

$$\psi(r)=\int_0^{+\infty}\theta(\eta)\psi(\eta)G_m(r,\eta;\sqrt{\lambda})\eta d\eta, \tag{8}$$

where G_m is defined by Eq. (7) and θ is given by Eq. (6).

We solve the nonlinear integral equation (8) using the stabilized relaxation procedure similar to that employed in Ref. [13]. Figure 1 shows several examples of the solutions of the system (5) found numerically for different values of the nonlocality parameter α. To characterize these solutions quantitatively, we define the effective radii r_ψ and r_θ of the intensity distribution $|\psi|^2$ and the temperature distribution θ, respectively, as follows: $r_\psi^2=P^{-1}\int r^2|\psi(r)|^2d^2\mathbf{r}$, $r_\theta^2=\int r^2\theta(r)d^2\mathbf{r}/\int\theta(r)d^2\mathbf{r}$. Figure 2(a) shows the radii r_ψ and r_θ as functions of the nonlocality parameter α. Both r_ψ and r_θ decrease monotonically when the nonlocality parameter grows. In the local limit ($\alpha\gg 1$), both r_ψ and r_θ saturate at the same finite value, which grows with the topological charge. Figure 2(b) shows the beam power $P=\int|E|^2d^2\mathbf{r}$ as a function of the nonlocality parameter α.

The important information on stability of the vortex solitons can be obtained from the analysis of small perturbations of the stationary states. The basic idea of such a linear stability analysis is to represent a linear perturbation as a superposition of the modes with different azimuthal symmetry.

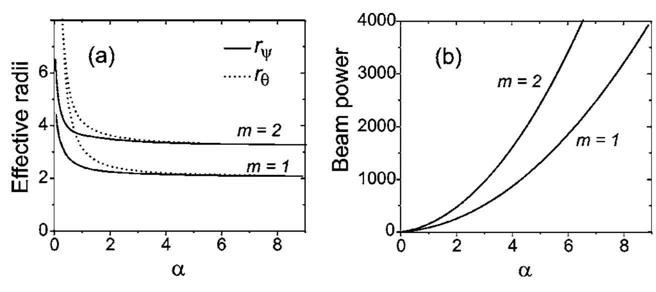

FIG. 2. (a) Effective radii of the field intensity distribution r_ψ (solid line) and the temperature field r_θ (dotted line) vs the nonlocality parameter α, for $m=1$, 2. (b) Power P vs α.

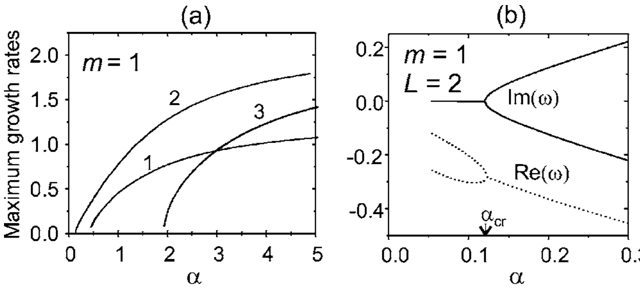

FIG. 3. Maximum growth rate of linear perturbation modes vs the nonlocality parameter α for the vortices with (a) $m=1$ and (c) $m=2$. The numbers near the curves stand for the azimuthal mode numbers L. (b) Real and imaginary parts of the eigenvalue ω of the most dangerous azimuthal mode with $L=2$ and $m=1$.

Since the perturbation is assumed to be small, stability of each linear mode can be studied independently. Presenting the nonstationary solution in the vicinity of the stationary state as follows,

$$\Psi(\mathbf{r},z) = \{\psi(r) + \delta\psi\}e^{i\lambda z + im\varphi}; \quad \Theta(\mathbf{r},z) = \theta(r) + \delta\theta,$$

$$\delta\psi = \varepsilon_+(r)\Phi + \varepsilon_-^*(r)\Phi^*, \quad \delta\theta = \vartheta_+(r)\Phi + \vartheta_-^*(r)\Phi^*,$$

where $\Phi(\varphi,z)=e^{i\omega z+iL\varphi}$, $|\varepsilon_\pm| \ll \psi$, $|\vartheta_\pm| \ll \theta$, ψ, θ are assumed to be real without loss of generality, we linearize Eqs. (4) and obtain the system of linear equations of the form

$$\pm\{-\lambda + \Delta_r^{(m\pm L)} + \theta(r) + \hat{g}_L\}\varepsilon_\pm \pm \hat{g}_L\varepsilon_\mp = \omega\varepsilon_\pm, \tag{9}$$

where

$$\hat{g}_L\varepsilon_\pm = \psi(r)\int_0^\infty \xi\psi(\xi)G_L(r,\xi;\alpha)\varepsilon_\pm(\xi)d\xi.$$

The Hankel spectral transform has been applied to reduce the integrodifferential eigenvalue problem (9) to linear algebraic equations. The maximum growth rate $|\mathrm{Im}\,\omega|$ of the linear perturbation modes is presented in Fig. 3(a) for a single-charge vortex ($m=1$). The symmetry-breaking modes can become unstable only for $L=1, 2, 3$. All growth rates saturate in the local regime, i.e., for $\alpha \gg 1$. The largest growth rate as well as the widest instability region has the azimuthal mode with the number $L=2$. The real and imaginary parts of the eigenvalues ω for this most dangerous mode are shown in Fig. 3(b). Importantly, there exists *a bifurcation point* $\alpha_{\mathrm{cr}} \approx 0.12$ below which the growth rate $\mathrm{Im}(\omega)$ vanishes. Thus, the symmetry-breaking azimuthal instability is eliminated in a highly nonlocal regime: all growth rates vanish provided $\alpha < \alpha_{\mathrm{cr}}$.

We also perform the linear stability analysis for multicharge vortices with the topological charges $m=2$ and $m=3$. Figure 3(c) shows the growth rate of the linear perturbation modes for the vortex with the charge $m=2$. Importantly, the growth rate of the $L=2$ mode remains nonzero, and the same result holds for the vortices with the charge $m=3$. Therefore, the linear stability analysis predicts the existence of stable single-charge vortices in a highly nonlocal regime, while the multicharge vortices remain unstable in this model with respect to a decay into the fundamental solitons, even in the limit $\alpha \to 0$.

Our linear stability analysis has been verified by direct simulations of the propagation dynamics of perturbed vortex solitons by employing the split-step Fourier method to solve Eqs. (4) numerically. The results of our linear stability analysis agree well with the numerical simulations. In particular, the symmetry-breaking instabilities have been observed in the region predicted by the linear stability analysis, and some examples of the vortex decay instability are presented in Fig. 4 for the vortices with the charge $m=1$. If a perturbation is applied to a single-charge vortex in the strongly nonlocal regime (such that all azimuthal instabilities are completely suppressed by the nonlocality), the input vortex beam evolves in a quasiperiodic fashion: the effective radii and amplitudes oscillate with z. Thus, our numerical simulations indicate that single-charge vortex solitons become stable if the nonlocality parameter α is below some critical value which is found to be very close to the value $\alpha_{\mathrm{cr}} \approx 0.12$ predicted by the linear stability analysis. We notice that the estimated values of the nonlocality parameter $\alpha \approx 10^{-3} \div 10^{-2}$ for the media with thermal nonlinearity are well within the stability region.

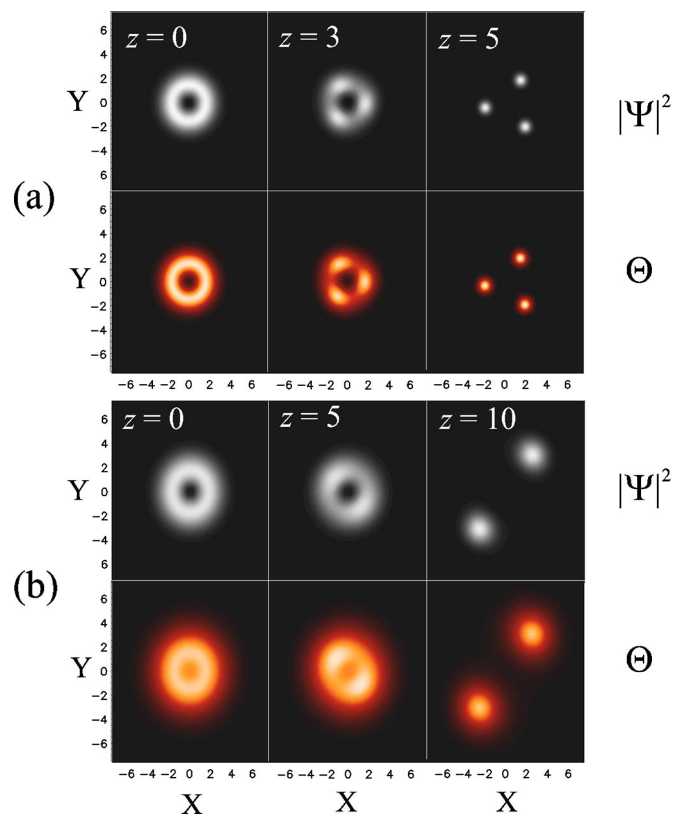

FIG. 4. (Color online) Evolution of the beam intensity $|\Psi|^2$ (upper rows) and temperature field θ (lower rows) of a perturbed single-charge vortex for (a) $\alpha=5$ and (b) $\alpha=1$.

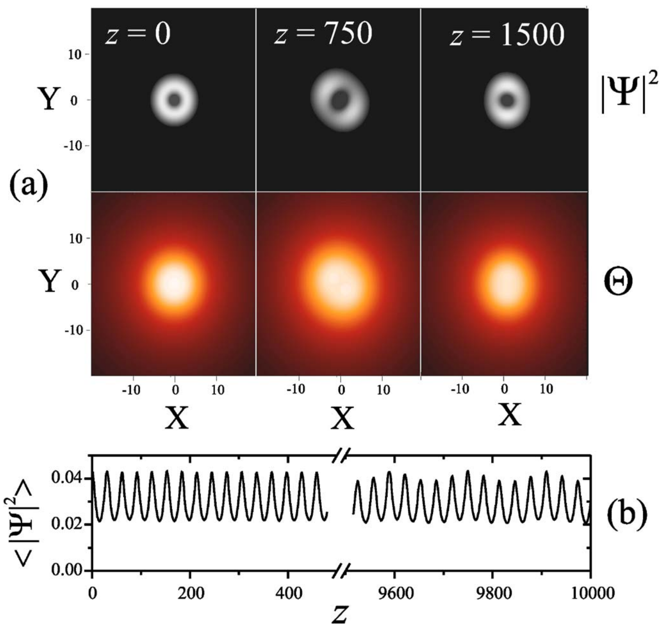

FIG. 5. (Color online). (a) Stable propagation of a vortex generated by a singular Gaussian beam with $m=1$ $(h=0.14, w=4.2, \alpha=0.07)$ shown for the beam intensity $|\Psi|^2$ and temperature field θ. (b) Oscillatory dynamics of the amplitude of the intensity field $\langle|\Psi|^2\rangle$ vs z during the beam propagation.

In experiment, the input beam may differ essentially from the exact vortex solution. Therefore, we perform additional numerical simulations for singular Gaussian input beams of the form $\Psi(r,0)=hr\exp(-r^2/w^2+i\varphi)$ and, as can be seen from Fig. 5, we observe that such beams can indeed propagate stably provided the nonlocality parameter is below the critical value; however, the beam effective intensity, defined as $\langle|\Psi|^2\rangle=P^{-1}\int|\Psi|^4 d^2\mathbf{r}$, undergoes large-amplitude oscillations.

The physical mechanism for suppressing the symmetry-breaking azimuthal instability of the vortex beam in a nonlocal nonlinear medium can be understood as being associated with *an effective diffusion process* introduced by a nonlocal response. Indeed, if a small azimuthal perturbation of the radially symmetric vortex beam deforms its shape in some region, the corresponding temperature distribution along the vortex ring becomes nonuniform. As a result, the intrinsic thermodiffusion process would smooth out this inhomogeneity and suppress its further growth, leading to the complete vortex stabilization in a highly nonlocal regime.

In conclusion, we have studied the basic properties and stability of spatially localized vortex beams in a self-focusing nonlinear medium with a nonlocal response. We have found that single-charge optical vortices become stable with respect to any symmetry-breaking azimuthal instability in the regime of strong nonlocal response, whereas multi-charged vortices remain unstable and decay into the fundamental solitons for any degree of nonlocality. Both the linear stability analysis and numerical simulations confirm stable propagation of single-charge vortices in a nonlocal nonlinear medium such as media with thermal nonlinearity and nematic liquid crystals. We expect that these results will stimulate the experimental observation of stable vortices created by coherent light in self-focusing nonlinear media.

Note added. We recently became aware of a related work by Briedis *et al.* [14] which demonstrated the vortex stabilization due to nonlocal nonlinearity in the framework of a different model.

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